1

Network calculus

Jean-Yves Le Boudec
Patrick Thiran
LCA-ISC, I&C EPFL

CH-1015 Lausanne
Patrick.Thiran@epfl.ch
http://icawww.epfl.ch

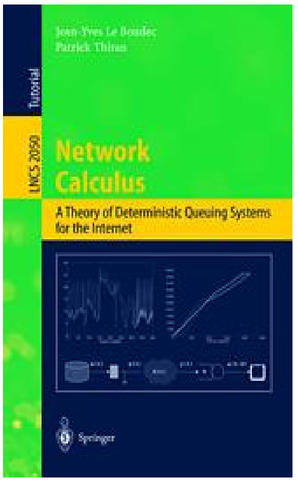
2

Contents

0. What is Network Calculus ?

"Network calculus", J-Y Le Boudec and P. Thiran, Lecture Notes in Computer Sciences vol. 2050, Springer Verlag, also available on-line at http://lcawww.epfl.ch

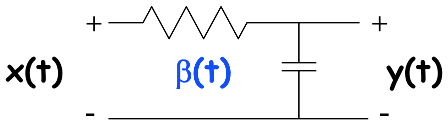
1. Arrival curves
2. Service curves, backlog, delay bounds
3. Playback delay for pre-recorded video



0. What is Network Calculus ?

- ☐ Deterministic analysis of queuing / flow systems arising in communication networks
- ☐ A Min-Plus or Max-Plus algebra filtering theory
- ☐ Some references
  - R-L. Cruz « A calculus for network delay, part I and part II », IEEE Trans. on Information Theory, pp. 114-141, Jan 1991.
  - C-S. Chang « Performance guarantees in Communication Networks », Springer-Verlag, New York, 2000.
  - J-Y. Le Boudec and P. Thiran « Network calculus», Lecture Notes in Computer Sciences vol. 2050, Springer Verlag, New York, 2000.

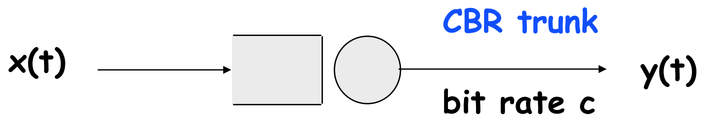
The standard Linear Theory



- ☐ A LTI filter in conventional algebra (R, +, ×)
  - Input signal = electrical voltage $x(t)$
  - System = circuit (filter) with impulse response $\beta(t)$
  - Output = convolution of $x(t)$ and $\beta(t)$:
    $y(t) = \int \beta(t-s)\, x(s)\, ds$

5

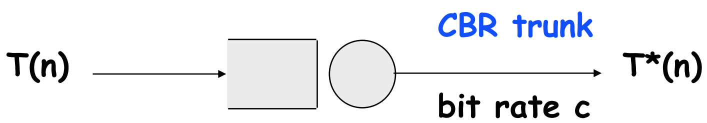
Network Calculus uses Min-Plus Linear Theory



- A linear system in min-plus algebra (R, min, +)
  - Input = arrived traffic in $[0,t]$, $x(t)$
  - System = CBR trunk of rate c : $\beta(t) = ct$
  - Output = convolution of $x(t)$ and $\beta(t)$:
    $y(t) = \inf_s \{\beta(t-s) + x(s)\}$

6

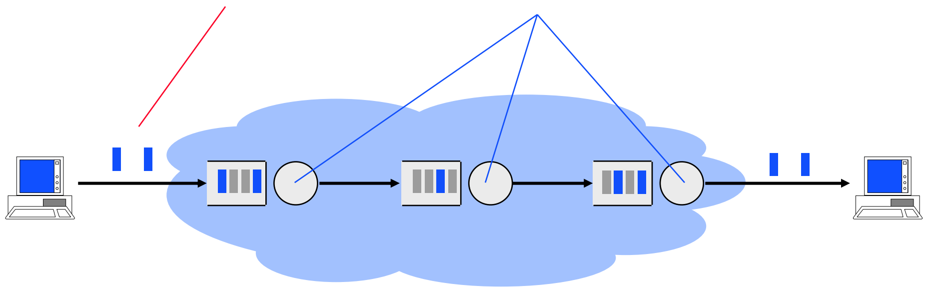
Network Calculus uses Max-Plus Linear Theory



- A linear system in max-plus algebra (R, max, +)
  - Input = arrival time of nth packet (cell): $T(n)$
  - System = CBR trunk of rate c: $\beta^{-1}(n) = 424n/c$
  - Output time = convolution of $T(n)$ and $\beta(n)$:
    $T^*(n) = \max_m \{\beta^{-1}(n-m) + T(m)\}$

Preliminary Concepts
Arrival and Service Curves

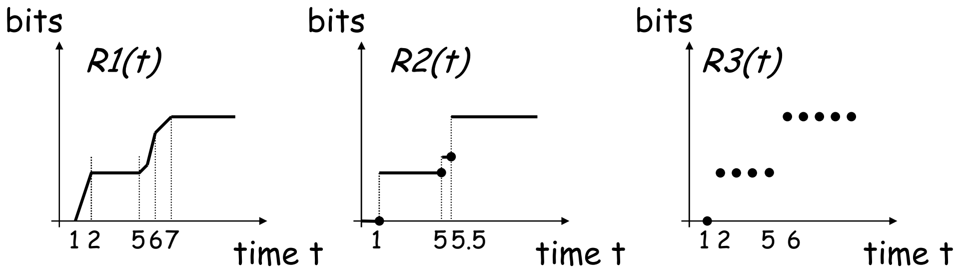
- Internet integrated services use the concepts of *arrival curve* and *service curves*



Contents

Cumulative flows

- Cumulative flow $R(t) \in \mathcal{F}$, t real or integer
- $\mathcal{F} = \{ x(t) \mid x(t)$ *is non decreasing and* $x(t) = 0$ *for* $t < 0 \}$
- **Examples:**



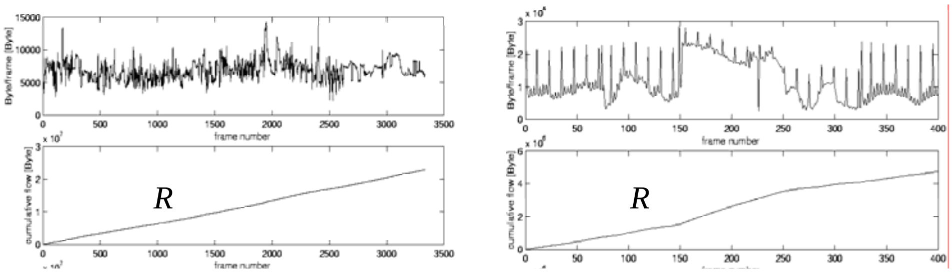
Fluid model (continuous)

Packet model (left continuous)

Discrete-time model

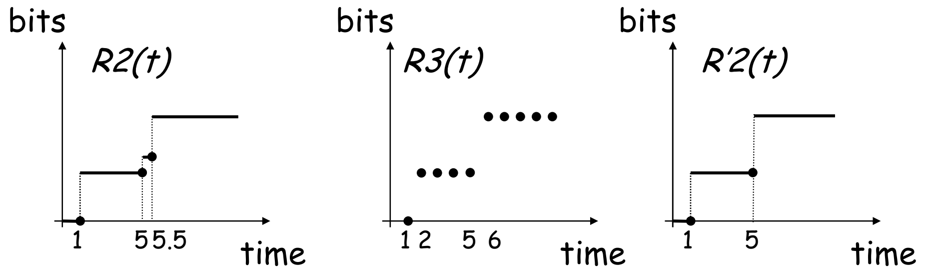
Example

- MPEG files, 25 frames/sec

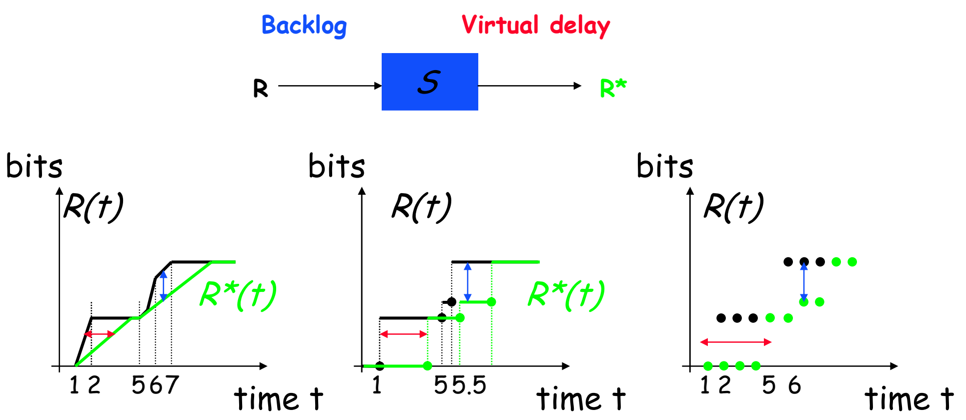


Cumulative flows

- $R3(n) = R2(n\delta)$, n integer, $\delta = 1$
- $R'2(t) = R3(\lceil t/\delta \rceil)$, t real, $\delta = 1$



Input and output flows



13

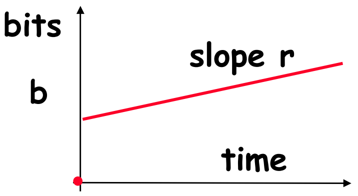
Arrival Curves

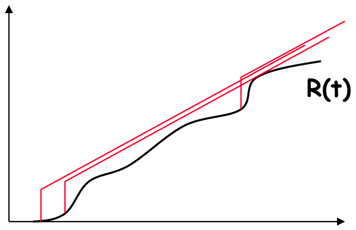
- Arrival curve α: For any times $0 \leq s \leq t$, the cumulative flow $R(.)$ satisfies

$$R(t) - R(s) \leq \alpha(t-s)$$

Example 1: affine arrival curve $\gamma_{r,b}$

$$\alpha(t) = \gamma_{r,b}(t) = rt+b \quad for\ t>0$$

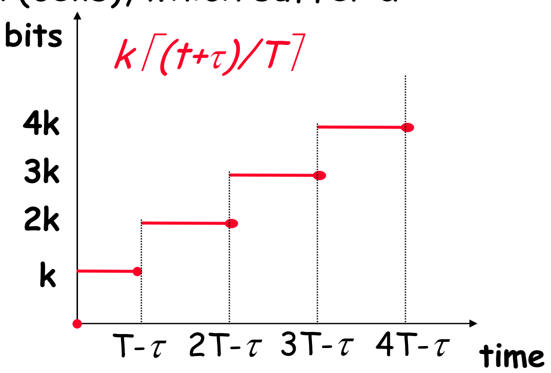




14

Example 2: stair arrival curve

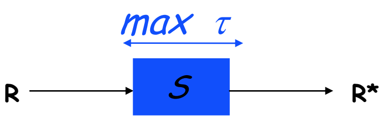
- $\alpha(t) = ku_{T,\tau}(t) = k\lceil (t+\tau)/T \rceil$ if $t > 0$
  with T = period, τ = tolerance
- Characterizes flows that are periodic stream of packets of same size k (cells), which suffer a variable delay <= τ



15

Example 2: stair arrival curve

- $\alpha(t) = ku_{T,\tau}(t) = k\lceil(t+\tau)/T\rceil$ with T = period, τ = tolerance
- Characterizes flows that are periodic stream of packets of same size k (cells), which suffer a variable delay <= τ

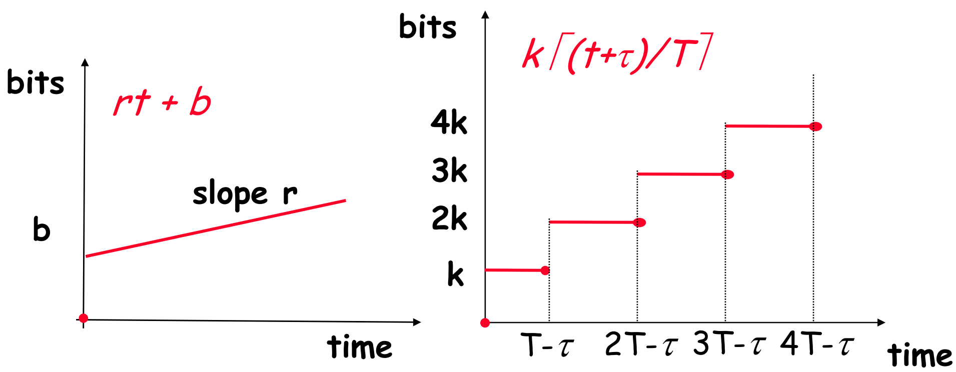


- Suppose that $R(t)$ is T-periodic: $R(t) - R(u) \leq k\lceil(t-u)/T\rceil$
- $R^*(s) \geq R(s-\tau)$
- $R^*(t) - R^*(s) \leq R(t) - R(s-\tau) \leq k\lceil(t-s+\tau)/T\rceil = ku_{T,\tau}(t-s)$

16

Arrival curve can assumed to be left continuous

- Technical lemma: if $\alpha(t)$ is an arrival curve for R, then $\alpha_l(t) = \sup_{s<t}\alpha(s)$ is also an arrival curve for R.

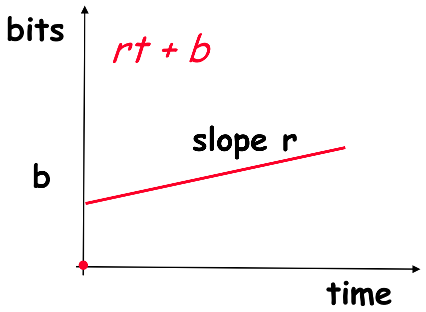


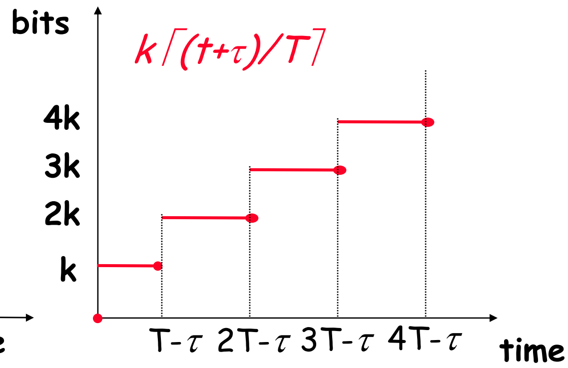
17

Affine and stair arrival curve

- $R(t)$ = flow of packets of **same** size k (cells)
- R conforms to $\alpha(t) = k u_{T,\tau}(t) = k \lceil (t+\tau)/T \rceil$

  ⇔ R conforms to $\alpha(t) = \gamma_{r,b}(t) = rt + b$ with $r = k/T$ and $b = k(\tau + T)/T$



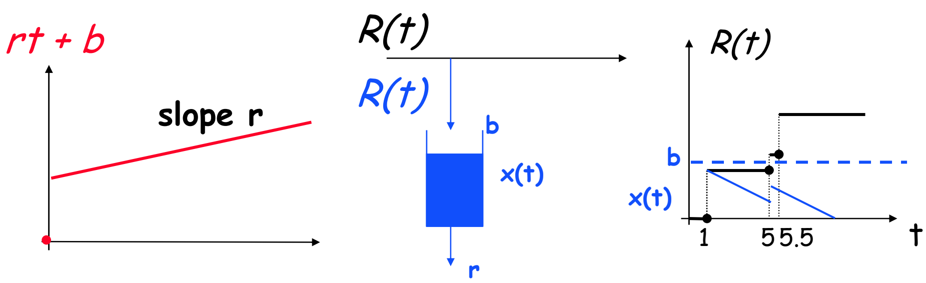


18

Leaky bucket

- All packets (cells) of flow R are declared conformant by a leaky buket controller of rate r and size b

  ⇔ R conforms to $\alpha(t) = \gamma_{r,b}(t) = rt + b$



Leaky bucket

- All packets (cells) of flow R are declared conformant by a leaky buket controller of rate r and size b

$\Leftrightarrow$ *R* conforms to $\alpha(t) = \gamma_{r,b}(t) = rt + b$

- ($\Rightarrow$) $(R(t)-x(t)) - (R(s)-x(s)) \leq r(t-s)$

  $\Rightarrow x(t) \geq R(t) - R(s) + x(s) - r(t-s) \geq R(t) - R(s) - r(t-s)$

  $\Rightarrow b \geq x(t) \geq R(t) - R(s) - r(t-s)$

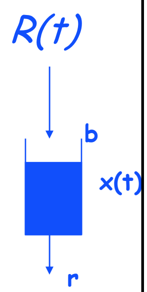
- ($\Leftarrow$) $R(t) - R(s) - r(t-s) \leq b$ for any s < t.

  Let *s* = beginning of busy period at time *t*: $x(s) = 0$.

  During $]s,t]$, the queue is never empty, so

  $x(t) = x(s) + R(t) - R(s) - r(t-s)$

  $\Rightarrow x(t) \leq b$



GCRA (T, τ)

- All packets (cells) of flow R are of the same size k
- Arrival time of nth = *A(n)*
- Theoretical arrival just after nth arrival is

  $\theta(n) = \max(A(n), \theta(n-1)) + T$

- If $A(n+1) >= \theta(n) - \tau$ then cell is conformant, otherwise not

  Example: GCRA (10,2)

n	1	2	3	3	4	5
$\theta(n-1)$	0	11	21	21	31	41
$A(n)$	1	11	16	20	29	38
	c	c	nc	c	c	nc

- Equivalences: *R* conforms to GCRA (T,τ)

$\Leftrightarrow$ *R* conforms to staircase arrival curve $\alpha = k u_{T,\tau}$

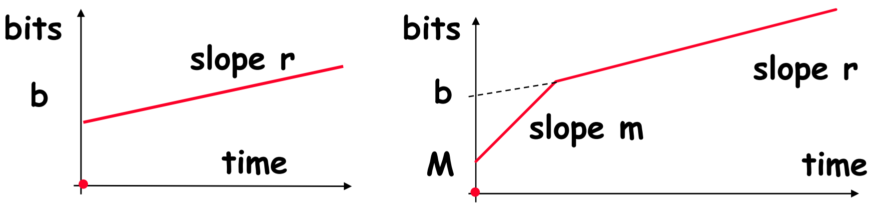
$\Leftrightarrow$ *R* conforms to leaky bucket ($r = k/T$, $b = k(\tau+T)/T$)

$\Leftrightarrow$ *R* conforms to affine arrival curve $\alpha = \gamma_{r,b}$

21

Combining leaky buckets

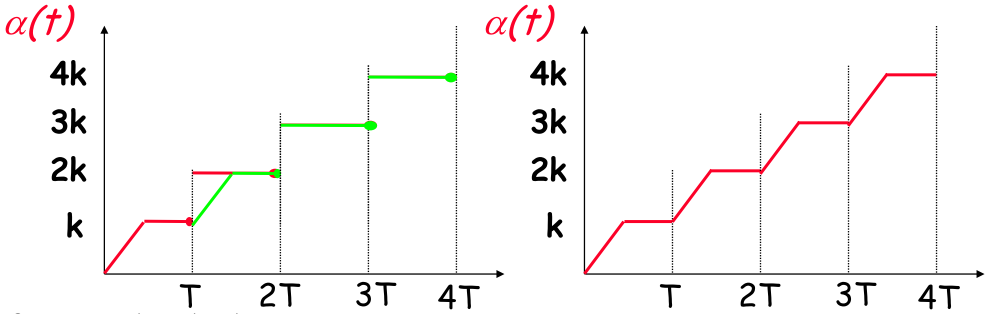
- standard arrival curve in the Internet ($\wedge$ = min)
  $\alpha(u) = \min\ (pu+M,\ ru+b) = (pu+M) \wedge (ru+b)$



22

Sub-additivity and arrival curves

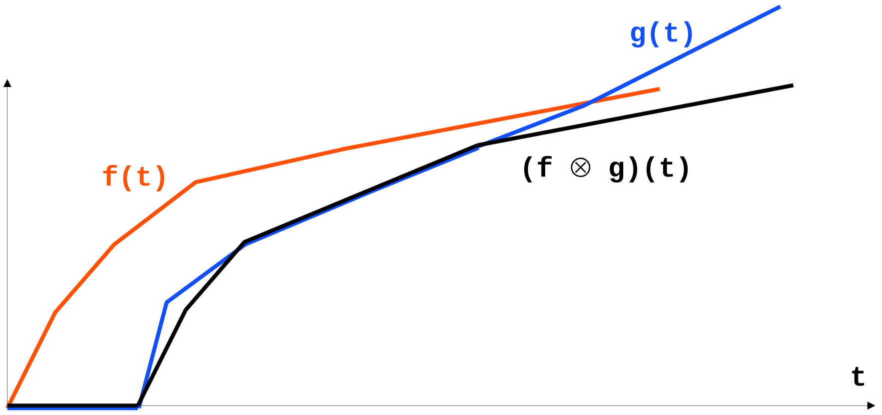
- If α is an arrival curve for flow R, so is $\bar{\alpha}$
- $\bar{\alpha}(t) \leq \alpha(t)$
- *What is* $\bar{\alpha}(t)$ *?*
- The answer uses min-plus convolution and sub-additivity



Min-plus convolution ⊗

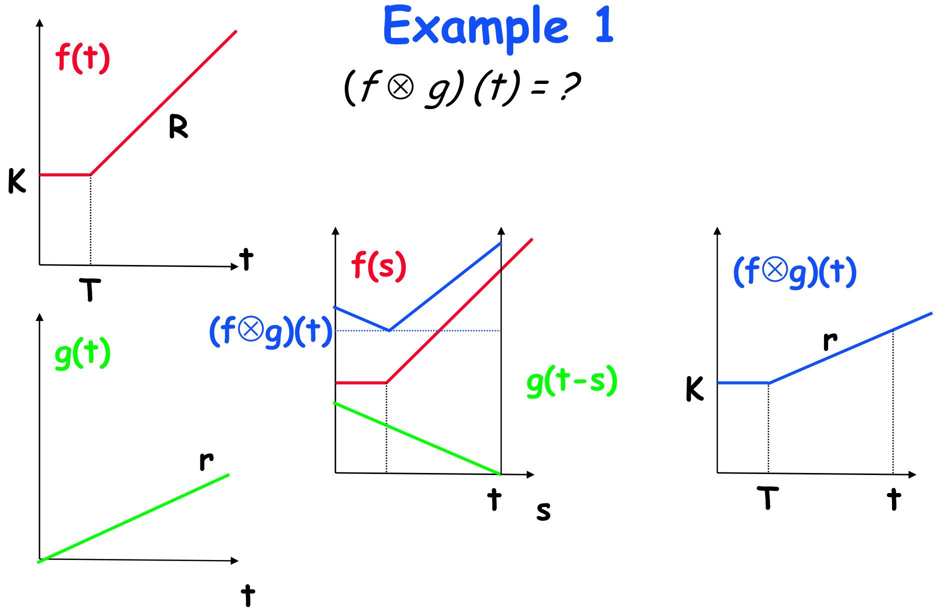
- Definition

$$(f \otimes g)(t) = \inf_u \{ f(t-u) + g(u) \}$$

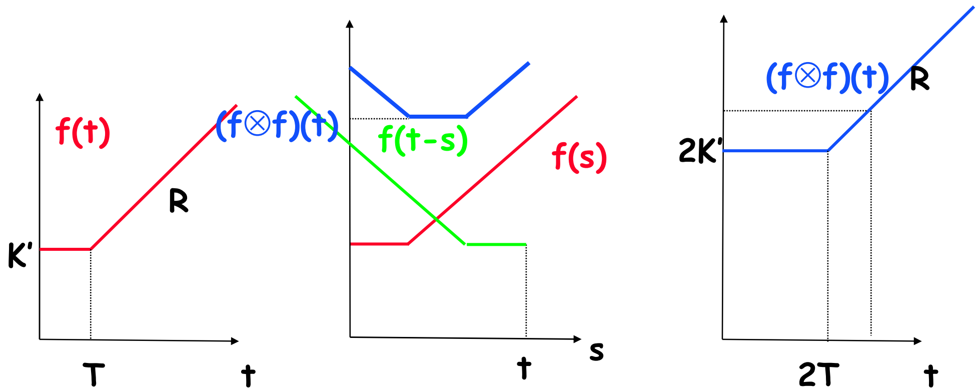


Example 1

$$(f \otimes g)(t) = ?$$



Example 2



$(f \otimes f)(t) = ?$

We can express arrival curves with min-plus convolution

- Arrival Curve property means for all $0 \le s \le t$,

$$x(t) - x(s) \le \alpha(t-s)$$

<-> $x(t) \le x(s) + \alpha(t-s)$ for all $0 \le s \le t$

<-> $x(t) \le \inf_u \{ x(u) + \alpha(t-u) \}$

<-> $x \le x \otimes \alpha$

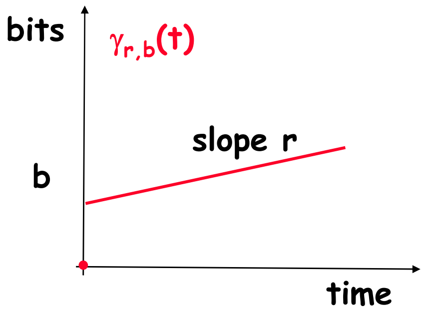
Star-shaped, concave, convex functions

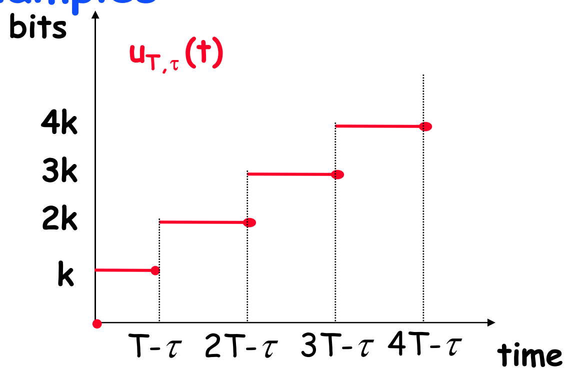
- $f \in \mathcal{F}$
- f is concave $\Leftrightarrow \forall\ 0 \leq u \leq 1$, $f(ux + (1-u)y) \geq uf(x) + (1-u)f(y)$
- f is convex $\Leftrightarrow$ $-f$ is concave
- f is star-shaped $\Leftrightarrow$ $f(t)/t \leq f(s)/s \quad \forall\ s \leq t$
- f is concave $\Rightarrow$ f is star-shaped
- f is star-shaped $\not\Rightarrow$ f is concave

Sub-additive function

- f is sub-additive $\Leftrightarrow$ $f(t) + f(s) \geq f(t+s)$
- f is concave with f(0) = 0 $\Rightarrow$ f is star-shaped
- f is sub-additive $\not\Rightarrow$ f is star-shaped
- f,g are sub-additive and pass through the origin (f(0) = g(0) = 0) $\Rightarrow$ $f \otimes g$ is sub-additive

Examples

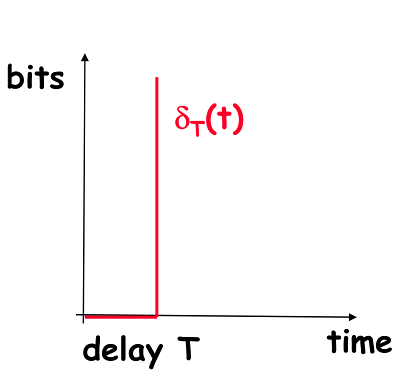


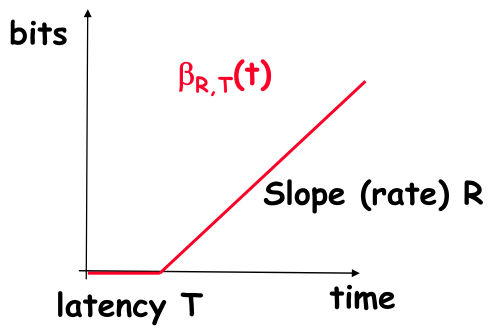


$\gamma_{r,b}$ is concave

$u_{T,\tau}$ is star-shaped ?
sub-additive ?

Examples

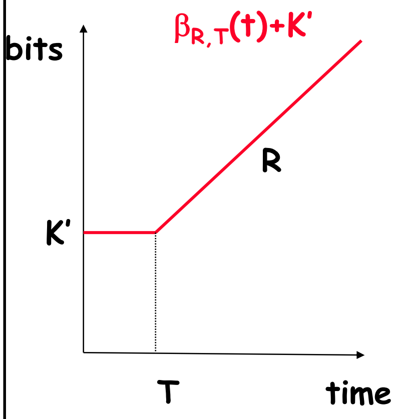


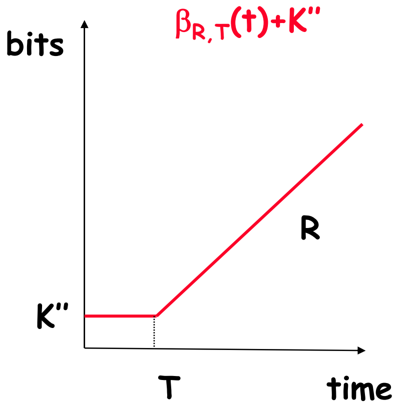


δ_T is convex

$\beta_{R,T}$ is convex

Examples





$\beta_{R,T} + K'$ is star-shaped ?
Sub-additive ?

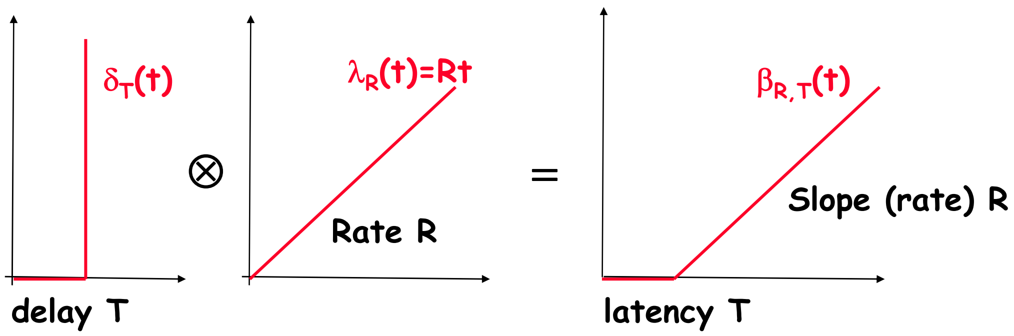
$\beta_{R,T} + K''$ is star-shaped ?
Sub-additive ?

Some properties of min-plus convolution

- $(f \otimes g) \in \mathcal{F}$
- $\otimes$ is associative
- $\otimes$ is commutative
- Neutral element: δ_0 : $f \otimes \delta_0 = f$
  ($\delta_0(t) = 0$ for $t = 0$ and $\delta_0(t) = \infty$ for $t > 0$)
- $\otimes$ is distributive with respect to $\wedge$
- $\otimes$ is isotone: $f \leq f'$ and $g \leq g' \Rightarrow f \otimes g \leq f' \otimes g'$
- Functions passing through the origin ($f(0) = g(0) = 0$):
  $f \otimes g \leq f \wedge g$
- Star-shaped (concave) functions passing through the origin:
  $f \otimes g = f \wedge g$
- Convex piecewise linear functions: $f \otimes g$ is the convex piecewise linear function obtained by putting end-to-end all linear pieces of f and g, sorted by increasing slopes

33

Example: rate latency function



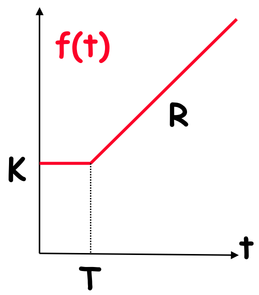
δ_T is convex (delay function)

λ_R is convex (delay function)

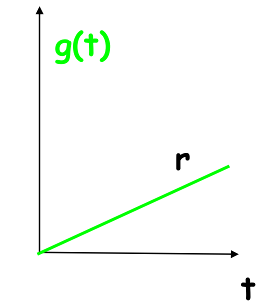
$\beta_{R,T}$ is convex (rate-latency function)

34

Example 1 (using rules)

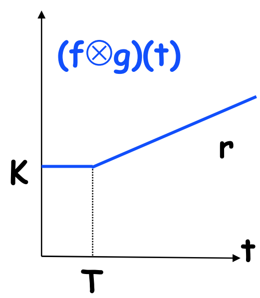


$f = K + \beta_{R,T}$
$= K + \delta_T \otimes \lambda_R$



$g = \lambda_r$ concave with $g(0) = 0$

$$
\begin{aligned}
f \otimes g &= (K + \delta_T \otimes \lambda_R) \otimes \lambda_r \\
&= K + ((\delta_T \otimes \lambda_R) \otimes \lambda_r) \\
&= K + (\delta_T \otimes (\lambda_R \otimes \lambda_r)) \\
&= K + (\delta_T \otimes (\lambda_R \wedge \lambda_r)) \\
&= K + (\delta_T \otimes \lambda_r) \\
&= K + \beta_{r,T}
\end{aligned}
$$



35

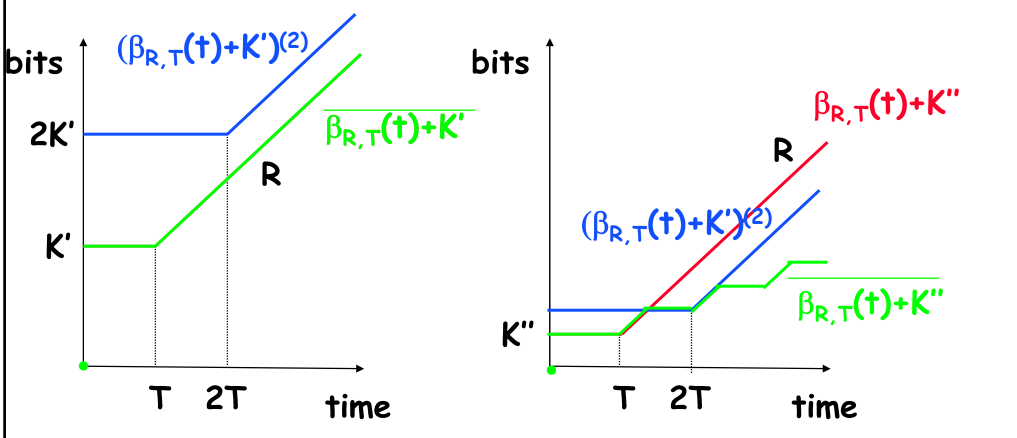
Sub-additive closure

- $\bar{f} = \inf \{\delta_0, f, f \otimes f, f \otimes f \otimes f, \ldots\}$
- $\bar{f}$ is sub-additive with f(0) = 0
- f is sub-additive with f(0) = 0 $\Leftrightarrow \bar{f} = f \Leftrightarrow f = f \otimes f$
- $f \le g \Rightarrow \bar{f} \le \bar{g}$
- $\overline{f \wedge g} = \bar{f} \otimes \bar{g}$
- Functions passing through the origin (f(0) = g(0) = 0):

$$\overline{f \otimes g} = \bar{f} \otimes \bar{g}$$

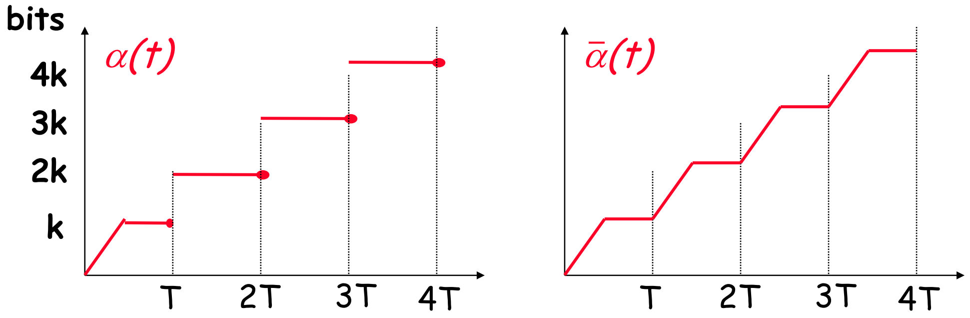
36

Examples



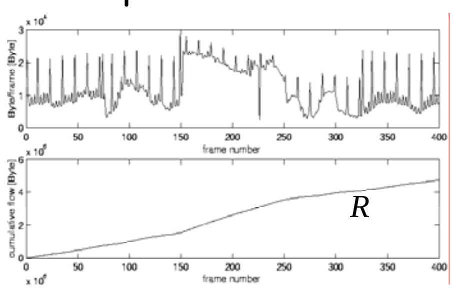
Sub-additivity and arrival curves

- What is $\bar{\alpha}(t)$?
- α can be replaced by its sub-additive closure $\bar{\alpha}$.
- From now on: we will always take sub-additive arrival curves passing through the origin.



Minimal arrival curve

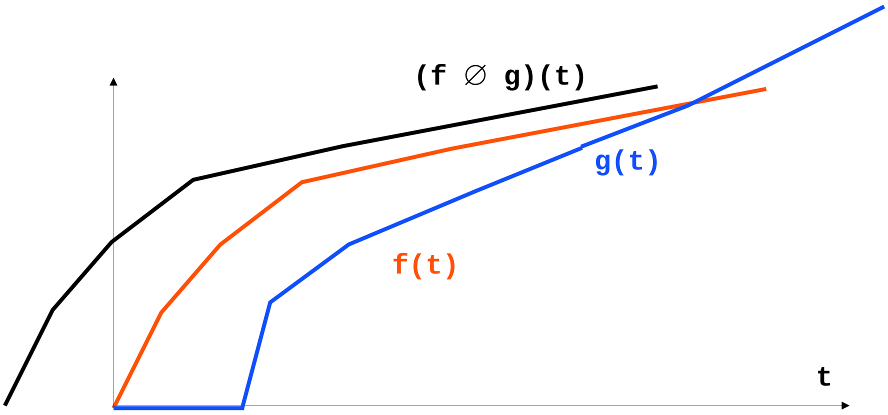
- If the only available information on a flow is obtained from measurements, i.e if we only know R, how can we compute its minimal arrival curve α?
- The answer uses min-plus deconvolution



Min-plus deconvolution $\oslash$

- Definition

$$(f \oslash g)(t) = \sup_u \{ f(t+u) - g(u) \}$$



Some properties of min-plus deconvolution

- $(f \oslash g) \notin \mathcal{F}$ in general
- $(f \oslash f) \in \mathcal{F}$
- $(f \oslash f)$ *is sub-additive with* $(f \oslash f)(0) = 0$
- $(f \oslash g) \oslash h = f \oslash (g \otimes h)$
- Duality with $\otimes$: $f \oslash g \leq h \Leftrightarrow f \leq g \otimes h$

Minimal arrival curve

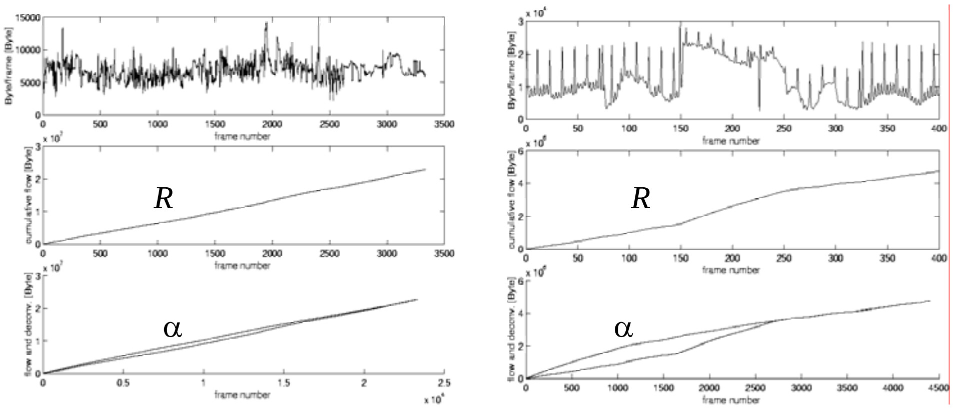
- The minimal arrival curve of flow R is $\alpha = R \oslash R$.
- Proof:
  - It is an arrival curve because

    $R(t) - R(s) = R((t-s)+s) - R(s)$

    $\leq \sup_u \{ R((t-s)+u) - R(u)\} = (R \oslash R)(t-s)$
  - If α' is another arrival curve for flow R, then $R \leq R \otimes \alpha'$ $\Leftrightarrow R \oslash R \leq \alpha'$ so that $\alpha \leq \alpha'$.

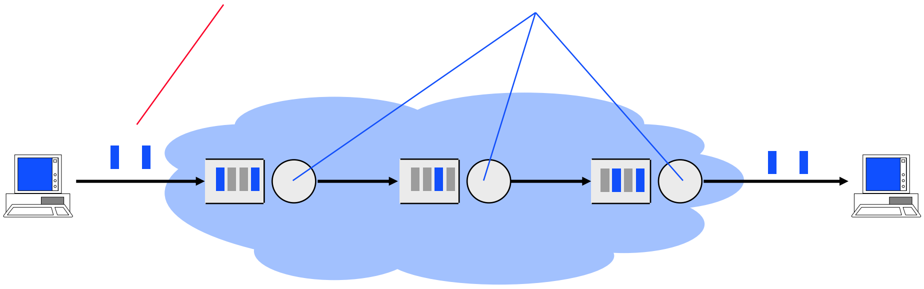
Example

- MPEG files, 25 frames/sec



Two key Concepts
Arrival and Service Curves

- Internet integrated services use the concepts of *arrival curve* and *service curves*



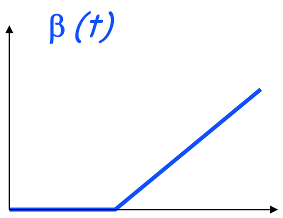
Contents

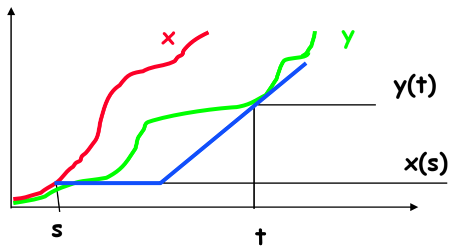
0. What is Network Calculus
1. Arrival curves
2. Service curves, backlog, delay bounds
   - Minimal vs strict service curves
   - Backlog and delay bounds
   - Concatenation
   - Packetizer
   - *Adaptive service curves*
   - *Guaranteed Rate Servers*
3. Playback delay for pre-recorded video

Minimal service Curve

- System S offers a (minimal) service curve β to a flow iff for all *t* there exists some *s* such that

$$y(t) - x(s) \geq \beta(t-s)$$



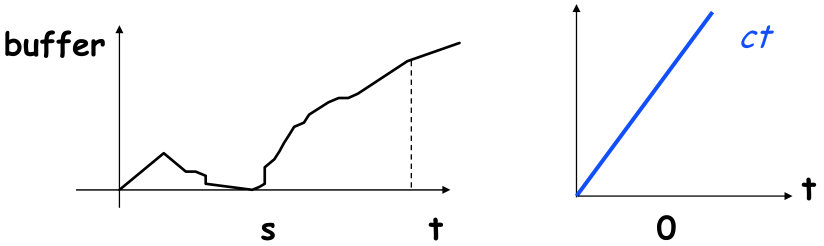


Strict service Curve

- Minimal service curve β : for all *t* there exists some *s* such that $y(t) - x(s) \geq \beta(t-s)$
- Strict service curve β : during any backlogged period [s,t], $y(t) - y(s) \geq \beta(t-s)$

47

The constant rate server has (strict) service curve $\beta(t)=ct$



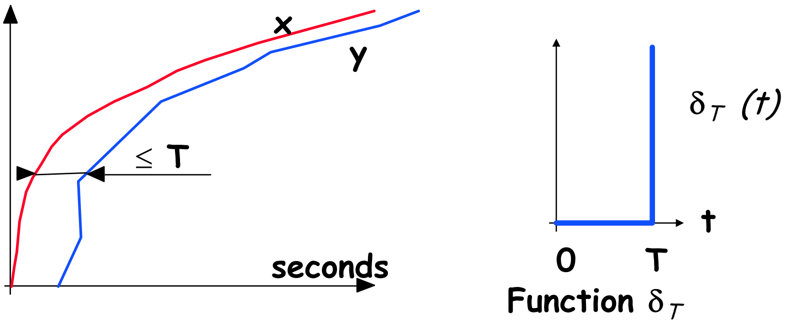
Proof: take s = beginning of busy period:

$y(t) - y(s) = c\,(t-s)$ and $y(s) = x(s)$

$\rightarrow\ y(t) - x(s) = c\,(t-s)$

48

The guaranteed-delay node has (minimal) service curve δ_T

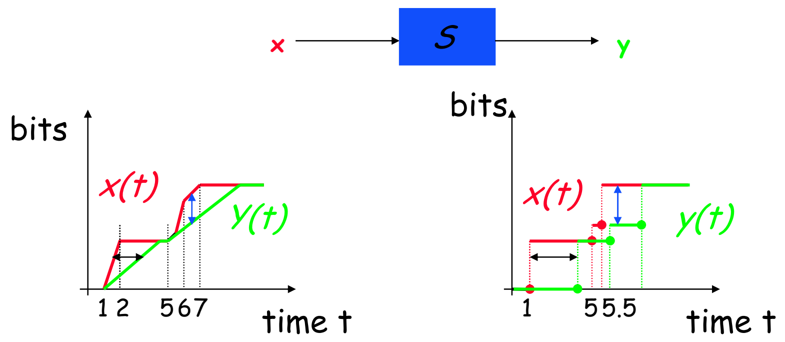


We can express service curves with min-plus convolution

- Service Curve guarantee means there exists some $0 \le s \le t$: $y(t) - x(s) \ge \beta(t-s)$
  - $\Leftrightarrow y(t) \ge x(s) + \beta(t-s)$ for some $0 \le s \le t$
  - $\Leftrightarrow y(t) \ge \inf_u \{ x(u) + \beta(t-u) \}$
  - $\Leftrightarrow y \ge x \otimes \beta$

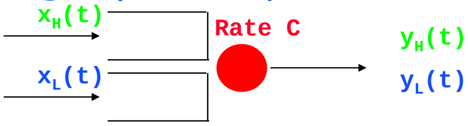
Backlog and (virtual) delay

- Flow x is constrained by arrival curve α
- System S offers a (minimal) service curve β to this flow
- Backlog at time t is $x(t) - y(t)$
- Virtual delay d(t) at time t is $d(t) = \inf\{ \delta \ge 0 \mid x(t) \le y(t+\delta) \}$



51

Non preemptive priority node – High priority traffic



- Pick any t, let s be the beginning of the busy period before t for HP traffic. With possible delay due to a LP packet that arrived just before s, of max size l_{max}

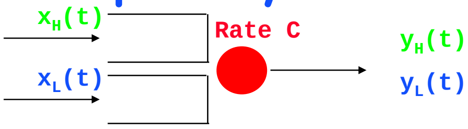
  $y_H(t) - y_H(s) \geq C(t-s) - l_{max}$ and $y_H(s) = x_H(s)$

  $\Rightarrow y_H(t) - x_H(s) \geq C(t-s) - l_{max}$
- Now, $y_H(t) - x_H(s) = y_H(t) - y_H(s) \geq 0$

  $\Rightarrow y_H(t) - x_H(s) \geq [C(t-s) - l_{max}]^+$
- Service curve for HP traffic is $\beta_{C, lmax/C}$

52

Non preemptive priority node Low priority traffic



- Assume α_H is an arrival curve for HP traffic.
- Let s' be the beginning of the busy period of the server before t.

  $y_L(t) - y_L(s') = C(t-s') - (y_H(t) - y_H(s'))$

  and $y_H(s') = x_H(s')$ and $y_L(s') = x_L(s')$

  $\Rightarrow y_L(t) - x_L(s') = y_L(t) - y_L(s') = C(t-s') - (y_H(t) - x_H(s'))$

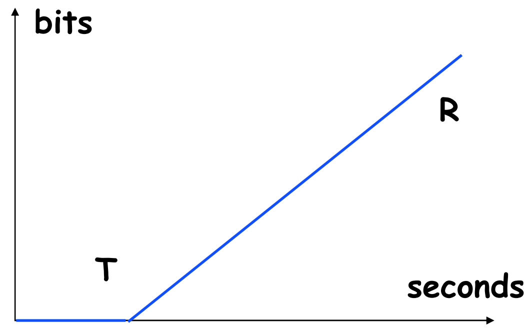
  $\geq C(t-s') - (x_H(t) - x_H(s')) \geq C(t-s') - \alpha_H(t - s')$
- Now, $y_L(t) - x_L(s') = y_L(t) - y_L(s') \geq 0$

  $\Rightarrow y_L(t) - x_L(s') \geq [C(t-s') - \alpha_H(t - s')]^+$
- Service curve for LP traffic is $S(t) = [Ct - \alpha_H(t)]^+$
- If $\alpha_H = \gamma_{r,b}$ then $S = \beta_{C-r, b/(C-r)}$

53

The standard model for an Internet router

- rate-latency service curve $\beta_{R,T}$

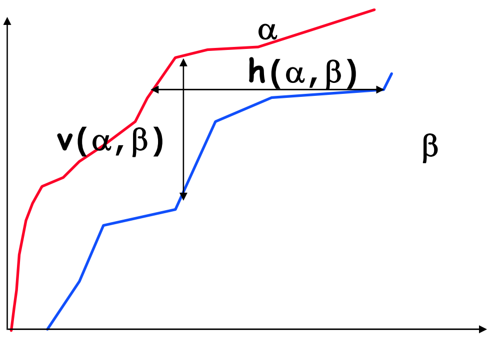


54

Three fundamental bounds

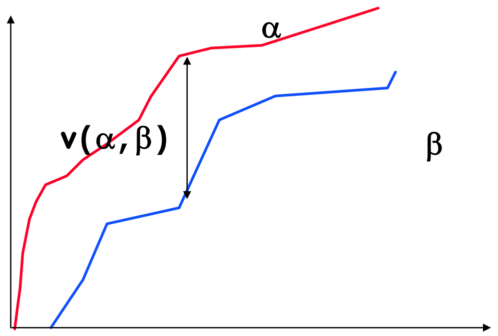
If flow has arrival curve α and node offers service curve β then

- backlog $\leq \sup (\alpha(s) - \beta(s)) = (\alpha \oslash \beta)(0) = v(\alpha, \beta)$
- delay $\leq \inf \{ s \geq 0 : (\alpha \oslash \beta)(-s) \leq 0 \} = h(\alpha, \beta)$
- Output flow y is constrained by arrival curve $\alpha' = \alpha \oslash \beta$



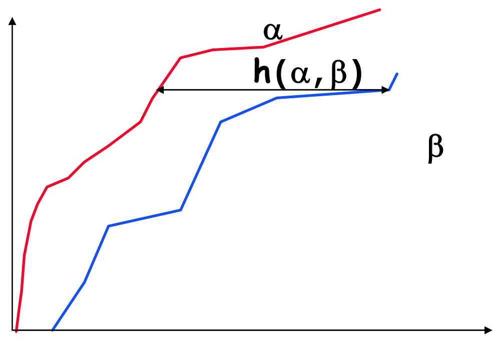
Tight Bound on backlog

- Flow x is constrained by arrival curve α
- System S offers a (minimal) service curve β to this flow
- Backlog at time t is $x(t) - y(t)$
- Backlog $\leq \sup (\alpha(s) - \beta(s)) = (\alpha \oslash \beta)(0) = v(\alpha, \beta)$



Tight Bound on delay

- Flow x is constrained by arrival curve α
- System S offers a (minimal) service curve β to this flow
- Virtual delay d(t) at time t is $d(t) = \inf\{ \delta \geq 0 \mid x(t) \leq y(t + \delta)\}$
- delay $\leq \inf \{ s \geq 0 : (\alpha \oslash \beta)(-s) \leq 0 \} = h(\alpha, \beta)$



57

Arrival Curve of output flow

- Flow x is constrained by arrival curve α
- System S offers a (minimal) service curve β to this flow
- Output flow y is constrained by arrival curve $\alpha' = \alpha \oslash \beta$

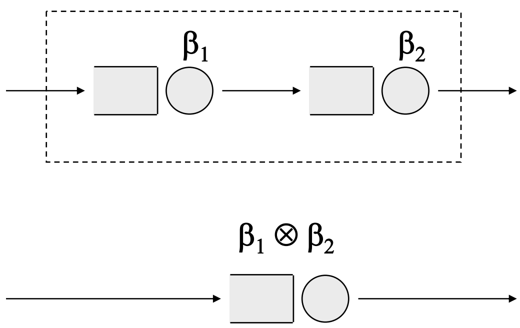
- Proof: $x(t) \leq y(t)$ and $y(s) \geq \inf_u (x(u) - \beta(s-u))$

$$\Rightarrow y(t) - y(s) \leq x(t) - \inf_u (x(u) - \beta(s-u))$$
$$= \sup_u \{x(t) - x(u) - \beta(s-u)\}$$
$$\leq \sup_u \{\alpha(t-u) - \beta(s-u)\}$$
$$= \sup_v \{\alpha(t-s+v) - \beta(v)\}$$
$$= (\alpha \oslash \beta)(t-s)$$

58

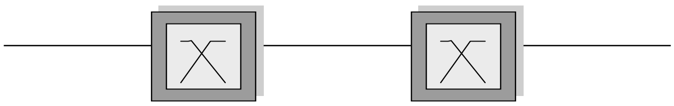
The composition theorem

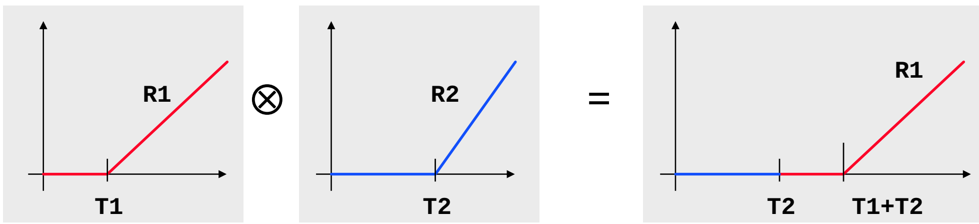
- **Theorem**: the concatenation of two network elements each offering service curve β_i offers the service curve $\beta_1 \otimes \beta_2$



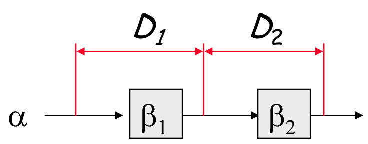
Example

- tandem of routers

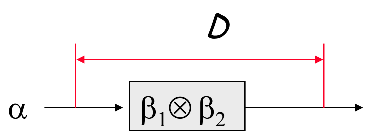




Pay Bursts Only Once



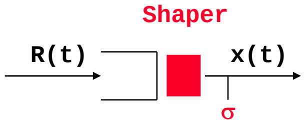
$$D_1 + D_2 \leq (2b + rT_1)/R + T_1 + T_2$$



$$D \leq b/R + T_1 + T_2$$

end to end delay bound is less

Greedy shaper



Definition of Greedy shaper

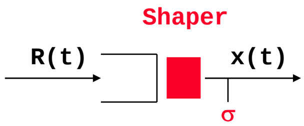
- forces output to be constrained by arrival curve σ

  $x(t) - x(s) \leq \sigma(t - s)$
- stores data in a buffer if needed
- Hence the shaper maximises $x(t)$ such that

  $x(t) \leq R(t)$

  $x(t) \leq (x \otimes \sigma)(t)$

Output of a Greedy shaper



- If σ is sub-additive and $\sigma(0) = 0$, $x(t) = (R \otimes \sigma)(t)$
- Proof:
  - $x = R \otimes \sigma$ is a solution because

    $x = R \otimes \sigma \leq R$ since $\sigma(0) = 0$

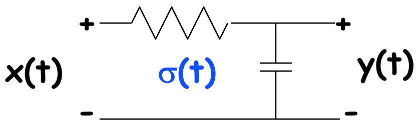
    $x = R \otimes \sigma = R \otimes (\sigma \otimes \sigma) = (R \otimes \sigma) \otimes \sigma = x \otimes \sigma$
  - If x' is another solution then $x' \leq R$ and $x' \leq x' \otimes \sigma$. Combining the two and using isotonicity of $\otimes$:

    $x' \leq x' \otimes \sigma \leq R \otimes \sigma = x$

Greedy shaper = linear min-plus filter

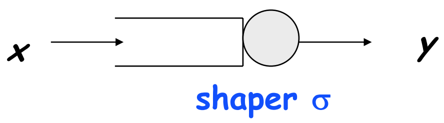
- Standard convolution in (R, x, +) (LTI filter)

$$y(t) = (\sigma * x)(t) = \int \sigma(t-u)\, x(u)\, du$$

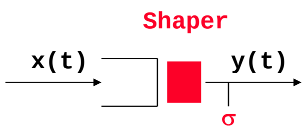


- Min-plus convolution in (R, +, ∧) is linear (∧ = min)

$$y(t) = (\sigma \otimes x)(t) = \inf_u \{ \sigma(t-u) + x(u) \}$$

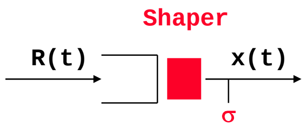


The service curve of a Greedy shaper is its shaping curve



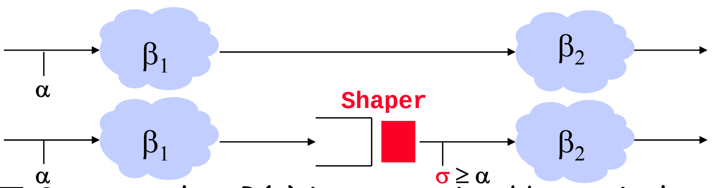
- If σ is sub-additive and $\sigma(0) = 0$, $y(t) = (x \otimes \sigma)(t)$.
- The service curve of a shaper is thus σ.

Shaping cannot be undone by shaping



- Suppose that R(t) is constrained by arrival curve $\alpha : R \leq R \otimes \alpha$.
- Then $x = R \otimes \sigma \leq (R \otimes \alpha) \otimes \sigma = R \otimes (\alpha \otimes \sigma) \leq R \otimes \alpha$ since $\sigma(0) = 0$.
- Therefore shaping keeps arrival constraints.
- In fact, the output flow has $\alpha \otimes \sigma$ as arrival curve

Re-shaping is for free



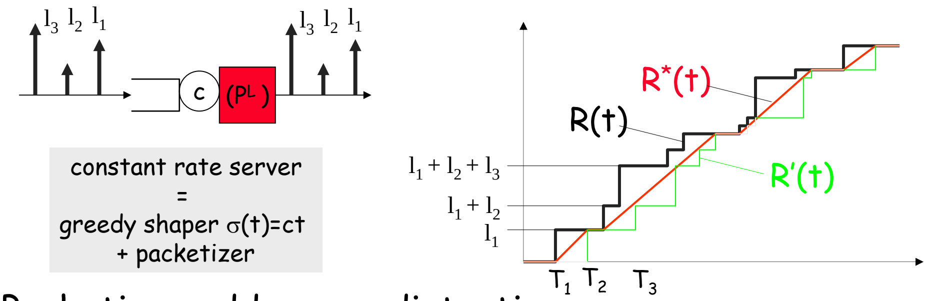
end-to-end service curve $\beta_1 \otimes \beta_2$

$\beta_1 \otimes \sigma \otimes \beta_2$

- Suppose that R(t) is constrained by arrival curve α
- Backlog for first system = $v(\alpha, \beta_1 \otimes \beta_2) = (\alpha \oslash (\beta_1 \otimes \beta_2))(0)$
- Backlog for second system with intermediate shaper =
  $v(\alpha, \beta_1 \otimes \sigma \otimes \beta_2) = (\alpha \oslash (\beta_1 \otimes \sigma \otimes \beta_2))(0) = (\alpha \oslash (\sigma \otimes \beta_1 \otimes \beta_2))(0)$
  $= ((\alpha \oslash \sigma) \oslash (\beta_1 \otimes \beta_2))(0)$
- *Since* $\alpha \leq \sigma$ and α is sub-additive with $\alpha(0) = 0$,
  $(\alpha \oslash \sigma)(t) = \sup_u\{\alpha(t+u) - \sigma(u)\} \leq \sup_u\{\alpha(t+u) - \alpha(u)\} = (\alpha \oslash \alpha)(t) = \alpha(t)$
- Therefore $v(\alpha, \beta_1 \otimes \sigma \otimes \beta_2) = (\alpha \oslash (\beta_1 \otimes \beta_2))(0) = v(\alpha, \beta_1 \otimes \beta_2)$
- Same reasoning for delays: $h(\alpha, \beta_1 \otimes \sigma \otimes \beta_2) = h(\alpha, \beta_1 \otimes \beta_2)$

Handling variable size packets

- The shaper presented before is for constant size packets or ideal fluid systems
- Real life systems are modelled by adding a packetizer
  - transforms fluid input into packets of size l_1, l_2, l_3, ...



- Packetizer adds some distortion

Goal of Service Curve and GR node definitions

- define an abstract node model
- independent of a specific type of scheduler
- applies to real routers, which are not a single scheduler, but a complex interconnection of delay and scheduling elements
- applies to nodes that are not work-conserving

Adaptive service Curve

- Minimal service curve β : for all t there exists some s such that $y(t) \geq \beta(t-s) + x(s)$
- Strict service curve β : during any backlogged period [s,t], $y(t) \geq \beta(t-s) + y(s)$
- Adaptive service curve β : for all t and all $s \leq t$,

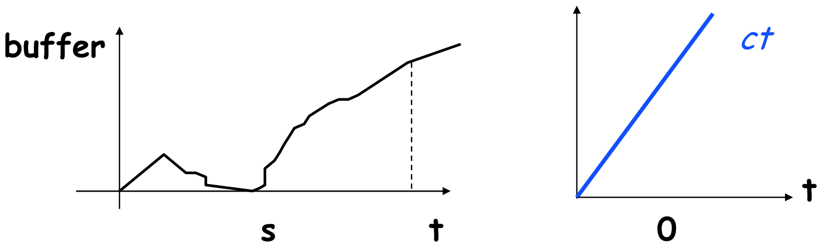
$$y(t) \geq \{\beta(t-s) + y(s)\} \wedge \inf_{s \leq u \leq t} \{\beta(t-u) + x(u)\}$$

(in fact: adaptive guarantee (β', β)

$$y(t) \geq \{\beta'(t-s) + y(s)\} \wedge \inf_{s \leq u \leq t} \{\beta(t-u) + x(u)\}$$

- Strict $\Rightarrow$ Adaptive $\Rightarrow$ Minimal

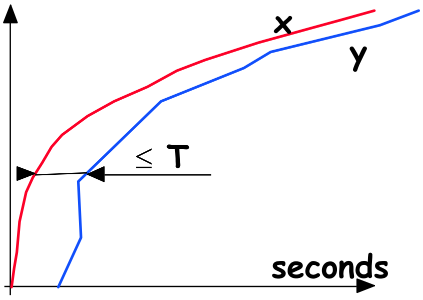
The constant rate server has (adaptive) service curve β(t)=ct

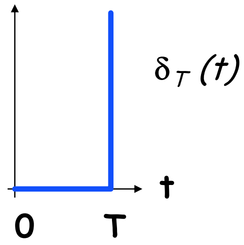


Proof: ct is a minimal service curve

71

The guaranteed-delay node has (adaptive) service curve δ_T



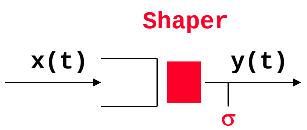


Proof: Pick any $s \leq t$.

- If $t-s \leq T$ then trivially $y(t) \geq \delta_T(t-s) + y(s)$
- If $t-s > T$ then $\inf_{s <= u <= t}\{\delta_T(t-u) + x(u)\} = x(t-T) \leq y(t)$

72

The adaptive service curve of a Greedy shaper is $(\sigma \oslash \sigma, \sigma)$



- Max-plus deconvolution: $(f \oslash g)(t) = \inf_u\{ f(t+u) - g(u)\}$
- $y(t) = (x \otimes \sigma)(t) = \inf_{0 <= u <= t}\{\sigma(t-u) + x(u)\}$

$$= \inf_{0 <= u \leq s}\{\sigma(t-u) + x(u)\} \wedge \inf_{s <= u <= t}\{\sigma(t-u) + x(u)\}$$

$$\geq \inf_{0 <= u \leq s}\{\sigma(s-u) + \inf_v\{\sigma(t-s+v) - \sigma(v)\} + x(u)\}$$
$$\wedge \inf_{s <= u <= t}\{\sigma(t-u) + x(u)\}$$

$$\geq \{ \inf_{0 <= u \leq s}\{\sigma(s-u) + x(u)\} + (\sigma \oslash \sigma)(t-s)\}$$
$$\wedge \inf_{s <= u <= t}\{\sigma(t-u) + x(u)\}$$

$$= \{ y(t) + (\sigma \oslash \sigma)(t-s)\} \wedge \inf_{s <= u <= t}\{\sigma(t-u) + x(u)\}$$

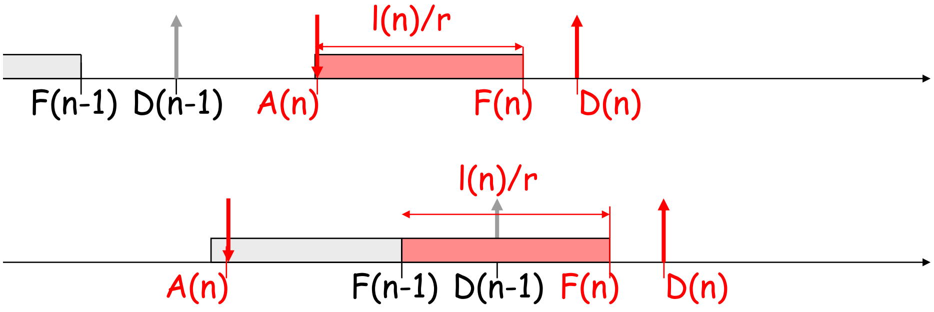
Guaranteed Rate node

- An alternative definition to service curve for FIFO
  - for rate-latency service curves only
- Definition (Goyal, Lam, Vin; Chang): a node is GR(r,e) if

$$D(n) \leq F(n) + e$$
$$F(n) = \max\{A(n), F(n-1)\} + l(n)/r$$

D(n) : departure time for packet n
A(n) : arrival time
F(n) : virtual finish time, F(0) = 0
l(n) : length in bits for packet n

F(n) = max{A(n), F(n-1)} A+ l(n)/r



GR is equivalent to rate-latency service curve -- for FIFO per flow

75

- GR(r,e) is equivalent to

  $D(n) \leq \max_{k \leq n}[A(k) + (l(k) + \ldots + l(n))/r] + e$

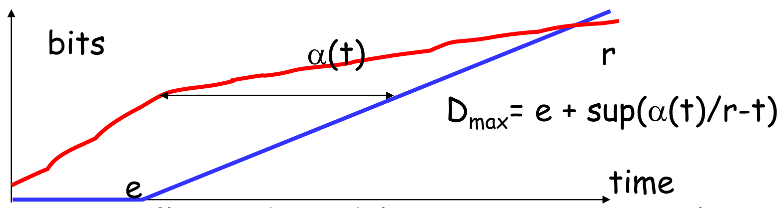
  - max-plus analog to service curve
- **Theorem (equivalence for FIFO per flow nodes)**:
  - a GR node is a service curve element with rate-latency service curve (r,e) followed by a packetizer
  - conversely, consider a node which is FIFO per flow and serves entire packets. If it has the rate-latency service curve (R,T) then it is GR(R,T).
- FIFO per flow is true in IntServ context

Properties of GR nodes (FIFO per flow or not)

76

- delay bound = $h(\alpha, \beta)$

  $D_{max} = e + \sup[\alpha(t)/r - t]$



for FIFO per flow nodes = delay at service curve element (packetizer does not add per-packet delay)

- backlog bound = $v(\alpha, \beta) + l_{max}$

  $B_{max} = \sup[\alpha(t) - R(t-T)^+] + l_{max}$

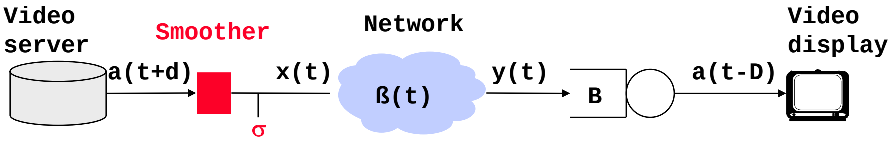
Modelling a node with GR

- queue with rate C: R=C, T=0
- priority queue with rate C: R=C, $T=l_{max}/C$
- element with bounded delay d: R = ∞, T=d
- and combine these elements

Contents

79

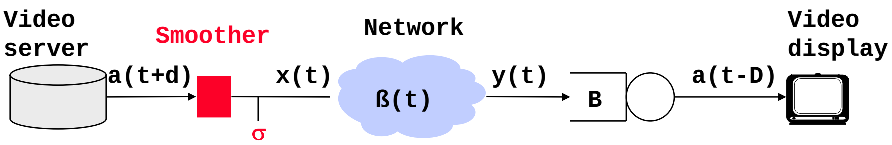
Network delivery of Pre-recorded video



- Le Boudec and Verscheure ToN 2000, Thiran, Le Boudec and Worm, Infocom 2001
- Network + end-station offers a service curve β to flow *x(t) (intserv or diffserv + real time model of end-station)*
- Smoother delivers a flow *x(t)* conforming to an arrival curve σ. Can look-ahead on the server (max *d* time units)
- Video stream is stored in the client buffer *B* and read after a playback delay *D*.

80

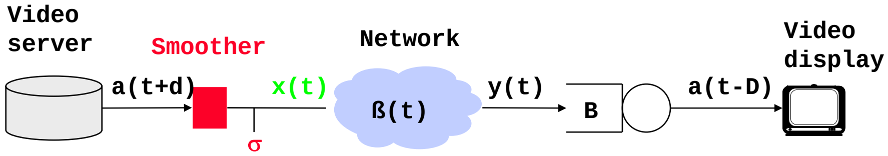
Network delivery of Pre-recorded video



- What are the minimal values of *D* and *B*, given *d*, σ and β ?
- What is the scheduling (smoothing) strategy at the sender side that achieves these minimal values ?
- Is this optimal smoothing strategy unique ?
- Does a large look-ahead delay *d* help in reducing *D* and *B* ?

81

Putting the Problem into Equations



- Smoothed flow $x(t)$ such that

  $x(t) \leq \delta_0(t)$ (i.e, $x(t) = 0$ if $t \leq 0$)

  $x(t) \leq a(t+d)$ (look-ahead up to d time units)

  $x(t) \leq (x \otimes \sigma)(t)$ (smoothing)

- Output flow $y(t)$ such that

  $y(t) \geq a(t-D)$ (no buffer underflow)

  $y(t) \leq a(t-D) + B$ (no buffer overflow)

- $y(t) = \Pi(x)(t)$ is not known but $(x \otimes \beta)(t) \leq y(t) \leq x(t)$

82

The Min-Plus Residuation Theorem

- From Baccelli et al, "Synchronization and Linearity"
- Theorem: Assume that operator Π is upper-semi-continuous. The problem

  $x(t) \leq a(t) \wedge \Pi(x)(t)$

  has one maximum solution, given by

  $x(t) = \underline{\Pi}(a)(t)$

- (Definition of closure of an operator)

  $\underline{\Pi}(x) = \inf\{x, \Pi(x), \Pi o \Pi(x), \Pi o \Pi o \Pi(x), \ldots\}$

- Π is upper-semi continuous if $\inf_i(\Pi(x_i)) = \Pi(\inf_i(x_i))$
  - true in practice for all our systems
- The greedy shaper output is an example of use

Massaging the Equations to use Residuation

- Output flow $y(t)$ such that
  $(x \otimes \beta)(t) \geq a(t-D)$ (no buffer underflow)
  $x(t) \leq a(t-D) + B$ (no buffer overflow)

  or equivalently using deconvolution operator ⊘
  $x(t) \geq (a \oslash \beta)(t-D) = \sup_u \{ a(t-D+u) - \beta(u) \}$
  $x(t) \leq a(t-D) + B$
- Therefore find smallest D, B s.t. maximal solution of
  $x(t) \leq \{ \delta_0(t) \wedge a(t+d) \wedge (a(t-D) + B) \} \wedge \{(x \otimes \sigma)(t)\}$

  verifies
  $x(t) \geq (a \oslash \beta)(t-D)$

Applying Residuation to our Problem

- Maximal solution of
  $x(t) \leq \{ \delta_0(t) \wedge a(t+d) \wedge (a(t-D) + B) \} \wedge \{(x \otimes \sigma)(t)\}$

  is, with σ sub-additive,
  $x(t) = \sigma \otimes \{ \delta_0(t) \wedge a(t+d) \wedge (a(t-D) + B) \}$
  $= \sigma(t) \wedge \{ (\sigma \otimes a)(t-D) + B \} \wedge (\sigma \otimes a)(t+d)$
- Need to check that this solution $x(t) \geq (a \oslash \beta)(t-D)$
  - $\sigma(t) \geq (a \oslash \beta)(t-D)$
    -> $D \geq h(a, \beta \otimes \sigma)$
  - $(\sigma \otimes a)(t-D) + B \geq (a \oslash \beta)(t-D)$
    -> $B \geq v(a \oslash a, \beta \otimes \sigma)$
  - $(\sigma \otimes a)(t+d) \geq (a \oslash \beta)(t-D)$
    -> $D + d \geq v(a \oslash a, \beta \otimes \sigma)$

Bounds for D, B and d

- In summary, we have shown that
  - the set of admissible playback delays D, playback buffer B and look-ahead limit d is

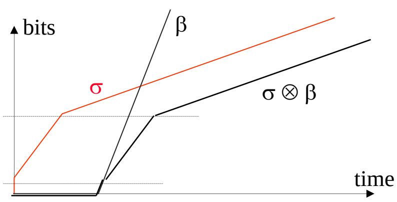
    $D \geq D_{min} = h(a, \beta \otimes \sigma)$

    $D + d \geq (D+d)_{min} = h(a \varnothing a, \beta \otimes \sigma)$

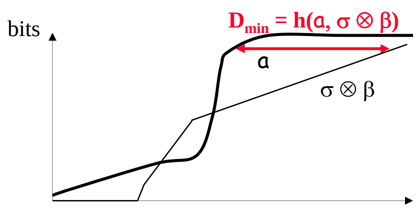
    $B \geq B_{min} = v(a \varnothing a, \beta \otimes \sigma)$
  - in particular, there is a minimum playback delay.
  - if D, d, B satisfy the constraints above, a schedule is possible;
    else, there is no schedule that can guarantee correct operation

The formulae have a simple graphical interpretation

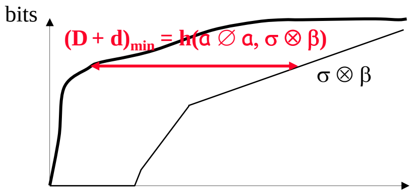
(1) compute $\sigma \otimes \beta$



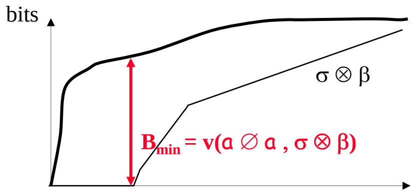
(2) compute the horizontal deviation



(3) compute $a \varnothing a$ and the horizontal deviation



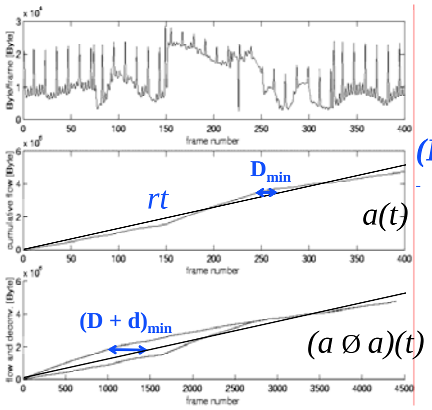
(4) compute the vertical deviation

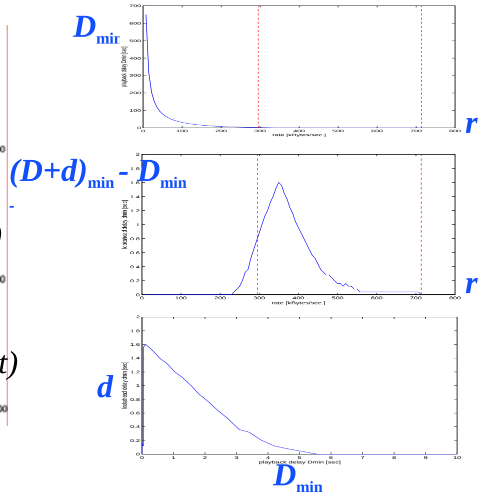


87

Example: MPEG Trace

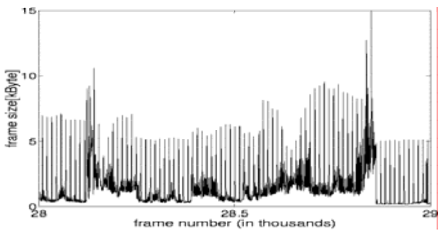
- MPEG files, 25 frames/sec, discretized in packets of 416 bytes

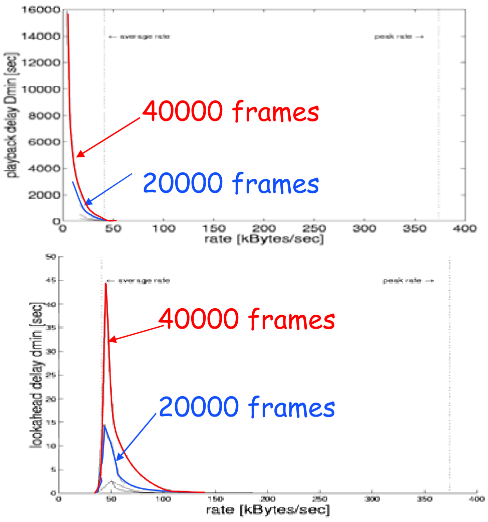




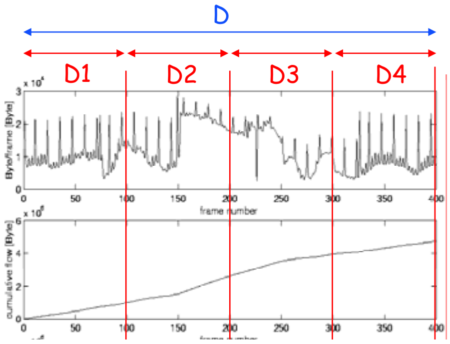
88

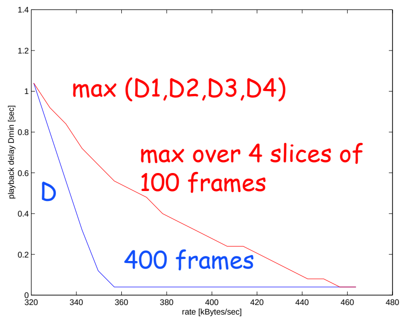
- Actual values of delays depend on the length of the stream and the position of largest burst, and the ability to predict it
  - Example: in Jurassic Park trace, largest burst occurs between frames 28000 - 29000





- Actual values of delays depend on the length of the stream, the position of largest burst, and the ability to predict it





Example 3: Dual problem formulation

- Find smallest D, B and d s.t. the maximal solution of

  $$x(t) \le \delta_0(t) \wedge R(t+d) \wedge \{R(t-D) + B\} \wedge (x \otimes \sigma)(t)$$

  verifies

  $$x(t) \ge (R \oslash \beta)(t-D).$$

- Property of $\oslash$: $x \le (x \otimes \sigma)$ <-> $(x \oslash \sigma) \le x$
- Find smallest D, B and d s. t. the minimal solution of

  $$x(t) \ge (R \oslash \beta)(t-D) \vee (x \oslash \sigma)(t)$$

  verifies

  $$x(t) \le \delta_0(t) \wedge R(t+d) \wedge \{R(t-D) + B\}.$$

91

Max-Plus System Theory in Action

- From Baccelli et al, "Synchronization and Linearity"; assume that Π is isotone and lower-semi-continuous.

  Theorem : the problem

  $$x(t) \leq a(t) \vee \Pi(x)(t)$$

  has one minimum solution, given by $x_{min}(t) = \overline{\Pi}(a)(t)$
- (Definition of super-additive closure)

  $$\overline{\Pi}(x) = \sup\{x, \Pi(x), \Pi o \Pi(x), \Pi o \Pi o \Pi(x), \ldots\}$$
- Minimal solution of

  $$x(t) \geq (R \oslash \beta)(t-D) \vee (x \oslash \sigma)(t)$$

  is, with σ sub-additive with $\sigma(0) = 0$,

  $$x_{min}(t) = (R \oslash (\beta \otimes \sigma))(t-D)$$

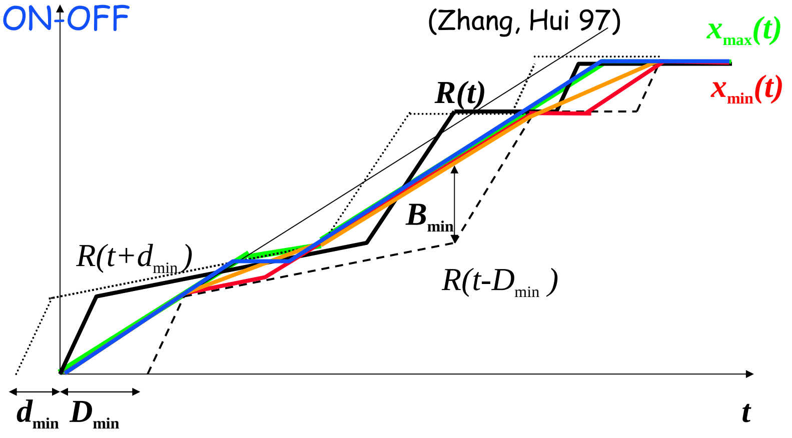
92

Scheduling for D_{min}, d_{min} and B_{min}

$x_{max}(t) = \sigma(t) \wedge (\sigma \otimes R)(t+d_{min}) \wedge \{(\sigma \otimes R)(t-D_{min}) + B_{min}\}$

$x_{min}(t) = (R \oslash (\beta \otimes \sigma))(t-D)$ (Le Boudec, Verscheure 2000)

+ Other metrics (Feng, Rexford 99):
+ minimal rate variability (Salehi, Zhang, Kurose, Towsley 98)
+ ON-OFF (Zhang, Hui 97)



Conclusion

- Network Calculus is a set of tools and theories for the deterministic analysis of communication networks
- A new system theory, which applies min-plus algebra to communication networks
- Does not supersede stochastic queueing analysis, but gives new tools for analysis of sample paths
- "Network calculus", J-Y Le Boudec and P. Thiran, Lecture Notes in Computer Sciences vol. 2050, Springer Verlag, also available on-line at `http://lcawww.epfl.ch`